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LETTER TO THE EDITOR

The Morse oscillator generalised from supersymmetry

E Drigo Filho

Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405-São Paulo, Brazil

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Abstract. Using the factorisation method in supersymmetric quantum mechanics we determine new potentials from the Morse oscillator. We apply this method although we do not use the ladder operators.

The number of systems in quantum mechanics having analytic solutions is limited. Thus, it is interesting to find more potentials from which we can determine the energy spectra. The factorisation method (FM) has provided one way to construct new potentials starting from known potentials. This method has been used in the harmonic oscillator (Mielnik 1984, Zhu 1987) and in the Coulomb potential (Fernandez 1984). Subsequently Nieto (1984) and Alves and Drigo Filho (1988) verified that the relation between the energy spectra of the different potentials is established through a superalgebra using supersymmetric quantum mechanics.

In this letter we apply the FM in the Morse potential. This potential, which has known solutions, was first written by Morse (1929) and treated more recently by Nieto and Simmons (1979) and Dahal and Sprigborg (1987). It has been used in chemical physics (e.g. ter Haar 1946) and in other branches of physics (e.g. Morse *et al* 1936).

Considering supersymmetric quantum mechanics with $N=2$ (Cooper and Freedman 1983, Ravndal 1984) we define the charges

$$Q = d\sigma_- = d\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1}$$

$$Q^+ = d^+\sigma_+ = d\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2}$$

where d and d^+ are bosonic operators, and we have the supersymmetric Hamiltonian:

$$H_{ss} = \{Q, Q^+\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} d^+d & 0 \\ 0 & dd^+ \end{pmatrix}. \tag{3}$$

H_- is called the supersymmetric partner of H_+. Both H_+ and H_- have the same spectrum except for the ground state, only H_+ has a normalisated ground state with eigenvalue $E_{+,0}=0$.

In order to determine new potentials we shall generalise the bosonic operators d and d^+.

On the other hand, the one-dimensional Morse potential given by Nieto and Simmons (1979) is

$$V_M(x) = D(1-e^{-ax})^2 \tag{4}$$

with the Schrödinger equation:

$$-\frac{d^2}{dy^2}\psi_+\lambda^2(1-e^{-y})^2\psi = \varepsilon_n\psi \tag{5}$$

where $\lambda = (2mD)^{1/2}/a\hbar$, $y = ax$ $(-\infty < y < +\infty)$, $\varepsilon_n = E_n/\mathscr{E}_0$ and $\mathscr{E}_0 = \hbar^2a^2/2m$.

The solution of (5) is

$$\psi_n(y) = N(\lambda, n)\exp[-y(\lambda - n - \tfrac{1}{2})]\exp(-\lambda e^{-y})L_n^{(2\lambda-2n-1)}(2\lambda e^{-y}) \tag{6}$$

$$\varepsilon_n = 2\lambda(n+\tfrac{1}{2}) - (n+\tfrac{1}{2})^2 \qquad n = 0, 1, \ldots [\lambda - \tfrac{1}{2}] \tag{7}$$

($[\lambda - \frac{1}{2}]$ being the greater integer smaller than $\lambda - \frac{1}{2}$) where $L_n^{(\alpha)}(x)$ are the associated Laguerre polynomials and $N(\lambda, n)$ is the normalisation constant.

The ladder operators in this case are

$$A_n^{\pm} = \frac{1}{2\lambda}\left((\lambda - n - \tfrac{1}{2})e^y \mp e^y\frac{d}{dy} - \frac{\lambda}{\lambda - (n+\frac{1}{2}) \mp \frac{1}{2}}\right) \tag{8}$$

with

$$A^{\pm}\psi_n \propto \psi_{n\pm1}. \tag{9}$$

From the Schrödinger equation (5) we obtain the Hamiltonian

$$H_+ = -d^2/dy^2 + \lambda^2(1-e^{-y})^2 - \lambda + \tfrac{1}{4}. \tag{10}$$

The constant term $-\lambda + \frac{1}{4}$ only displaces the spectrum. It makes the eigenvalue of the ground state equal to zero, $E_{+,0} = 0$.

This Hamiltonian can be factorised as

$$H_+ = a^+a \tag{11}$$

where

$$a = d/dy + \lambda(1-e^{-y}) - \tfrac{1}{2} \tag{12}$$

$$a^+ = -d/dy + \lambda(1-e^{-y}) - \tfrac{1}{2}. \tag{13}$$

We note, comparing (12) and (13) with (8), that these operators are not the creation and annihilation ones, and they satisfy the commutation relation

$$[a^+, a] = -2\lambda e^{-y}. \tag{14}$$

From this relation we can define a Hamiltonian, which is the supersymmetric partner of the Hamiltonian (10), inverting the operators

$$H_- = aa^+ = a^+a - [a^+, a] = a^+a + 2\lambda e^{-y} \tag{15}$$

corresponding to the potential

$$V_- = \lambda^2(1-e^{-y})^2 + 2\lambda e^{-y} - \lambda + \tfrac{1}{4}. \tag{16}$$

which is different from the original Morse potential.

H_+ and H_- being supersymmetric partners (see equation (3)), equations (15) and (10) have the same spectrum, but only (10) has a ground state with zero eigenvalue.

The eigenfunctions of (15) can be determined from the eigenfunctions ψ_+ of (10):

$$aH_+\psi_{+,n} = a\varepsilon_n\psi_{+,n} \Rightarrow H_-a\psi_{+,n} = \varepsilon_n a\psi_{+,n}. \tag{17}$$

Thus, the eigenfunctions $\psi_{-,n}$ of (15) are

$$\psi_{-,n} = a\psi_{+,n}. \tag{18}$$

To find the generalised potential we define new operators

$$A = \mathrm{d}/\mathrm{d}y + f(y) \tag{19}$$

$$A^+ = -\mathrm{d}/\mathrm{d}y + f(y) \tag{20}$$

and imposing that

$$H_- = AA^+ \tag{21}$$

we obtain the differential equation

$$\lambda^2(1-\mathrm{e}^{-y})^2 + 2\lambda\,\mathrm{e}^{-y} = \frac{\mathrm{d}}{\mathrm{d}y}f(y) + f^2(y) \tag{22}$$

which is a Ricatti differential equation having the solutions

$$f(y) = \lambda(1-\mathrm{e}^{-y}) - \tfrac{1}{2} + \frac{\exp[-y(2\lambda-1) - 2\lambda\,\mathrm{e}^{-y}]}{\Gamma + \int_0^y \exp[-\bar{y}(2\lambda-1) - 2\lambda\,\mathrm{e}^{-\bar{y}}]\,\mathrm{d}\bar{y}} \tag{23}$$

where the constant Γ is arbitrary and we have chosen $\Gamma > 0$ to avoid singularities.

The new operators satisfy the commutation relation

$$\begin{aligned}[A^+, A] &= -2\frac{\mathrm{d}}{\mathrm{d}y}f(y) \\ &= -2\lambda\,\mathrm{e}^{-y} - 2\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\exp[-y(2\lambda-1) - 2\lambda\,\mathrm{e}^{-y}]}{\Gamma + \int_0^y \exp[-\bar{y}(2\lambda-1) - 2\lambda\,\mathrm{e}^{-\bar{y}}]\,\mathrm{d}\bar{y}}\right) \\ &\equiv -2\lambda\,\mathrm{e}^{-y} - 2\phi'(y)\end{aligned} \tag{24}$$

and we can define a new Hamiltonian using the generalised operators (19) and (20):

$$\mathscr{H}_+ = A^+A = AA^+ + [A^+, A] = AA^+ - 2\lambda\,\mathrm{e}^{-y} - 2\phi'(y) \tag{25}$$

corresponding to the potential

$$\mathscr{V}_+ = \lambda^2(1-\mathrm{e}^{-y}) - \lambda + \tfrac{1}{4} - 2\phi'(y) \tag{26}$$

which is different from the original Morse potential and also from V_- given in (16).

From supersymmetry we know that the spectrum of $\mathscr{H}_+$ is the same as the spectrum of (15) (except the ground state) and the eigenfunctions of $\mathscr{H}_+$ are associated with the eigenfunctions of H_-:

$$A^+H_-\psi_{-,n} = A^+\varepsilon_n\psi_{-,n} \Rightarrow \mathscr{H}_+A^+\psi_{-,n} = \varepsilon_n A^+\psi_{-,n}. \tag{27}$$

This relation tells us that the $\mathscr{H}_+$ eigenfunctions are

$$\Psi_{+,n} = A^+\psi_{-,n} \tag{28}$$

and the ground state is given by

$$A\Psi_{+,0} = 0. \tag{29}$$

This means that

$$\Psi_{+,0} \propto \mathrm{e}^{-y/2} \exp[\lambda(y+\mathrm{e}^{-y})] \exp\left(\int_0^y \phi(\bar{y})\,\mathrm{d}\bar{y}\right). \tag{30}$$

The new eigenfunctions $\Psi_{+,n}$ are determined from the original Morse eigenfunctions (6). The map between these eigenfunctions is found by using the relations (18) and (28):

$$\Psi_{+,n} = A^{+} a \psi_{+,n}. \tag{31}$$

Thus, the operator $A^{+}a$ takes the Morse eigenfunctions and transforms them into $\mathscr{H}_{+}$ eigenfunctions. Note that the $\mathscr{H}_{+}$ ground state is not included in this map; it is given by (30).

The operator A^{+} alone permits us to map the eigenfunctions of (15) in the $\mathscr{H}_{+}$ ones (this is shown by relation (28)) and the operator A makes the inverse since

$$A\Psi_{+,n} = A(A^{+}\psi_{-,n}) = H_{-}\psi_{-,n} \propto \psi_{-,n}. \tag{32}$$

The multiplication constant can be absorbed by the normalisation constant. Then we see that A maps $\mathscr{H}_{+}$ eigenfunctions into H_{-} (15) eigenfunctions. The operators a and a^{+} play an analogous role between the Hamiltonians (10) and (15). These relations arise because the supersymmetric algebra is contained in the FM (Alves and Drigo Filho 1988).

We observe that, contrary to the usual FM, the operators that factorise the Hamiltonian (10) are not ladder operators. However, we can still construct the supersymmetric charges as defined in (1) and (2) and hence identify the bosonic operators d and d^{+} with a and a^{+} or A and A^{+}. Consequently we have the supersymmetric algebra which allows us to generalise the potential. We also note that we have to know the solution of the initial system, the Morse potential in this case, in order to obtain the new solutions.

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